Student seminar solutions Week 5

1. Let A be an open subset of $\mathbb{C}$ and let (f_n) be a sequence of holomorphic functions on A that converges uniformly on every compact subset to a function f.
   Show that f is holomorphic on A and that the sequence of derivatives (f'_n) converges uniformly on all compact subsets to f'.

   Proof. We let D be a closed disk contained in A and $C = \partial D$ be its boundary, oriented counter-clockwise.
   We may apply Cauchy's integral formula to deduce that given any $s_0 \in D \backslash C$ in the interior of D we get for all $n \in \mathbb{N}$:

   $$f_n(s_0) = \frac{1}{2\pi i} \int_C \frac{f_n(s)}{s - s_0} ds$$

   Since we have uniform convergence, we may pass to the limit to obtain

   $$f(s_0) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s - s_0} ds$$

   This shows that f is holomorphic on the interior of D and so it follows that f is holomorphic on U as desired.
   Let $K \subset A$ be a compact set, and since $A \neq \mathbb{C}$, we must have $d(K, \partial A) > 0$, say $d(K, \partial A) = d$. Consider $K' = \bigcup_{x \in K} D(x, d/2)$, the union of disks centered at points in K. We have inclusions $K \subset K' \subset A$. The set K' is further bounded, so it remains to how that K' is closed to conclude that it is compact. We do this by proving that the complement is open. Take $p \notin K'$. By compactness of K, there exists $q \in K$ that realises the infimum in the distance of p to K, i.e. $|p - d| = d(p, K)$. Since $p \notin K'$, we must have $d(p, K) > d/2$, and so $D(p, |p - q| - d/2)$ is an open disk around p that does not intersect K' and so K' is closed.
   Now, $\{f_n\}$ converges uniformly on K' by assumption. Let $\varepsilon > 0$ and find an N such that $|f_n(s) - f(s)| < \varepsilon$ for all $s \in K'$ and all $n \geq N$. For any $p \in K$, the disc $D = D(p, d/2)$ is fully contained in K' and has boundary in K', and so we may use the Cauchy Formula to obtain:

   $$|f'_n(p) - f'(p)| = |\frac{1}{2\pi} \int_{\partial D} \frac{f_n(t)}{(t-p)^2} - \frac{f(t)}{(t-p)^2} dt| \leq \frac{||f_n - f||_D}{(d/2)} \leq \frac{2\varepsilon}{d}$$

and so f_n' converges uniformly to f' on K.

□

2. Prove the following Lemma:

Lemma 0.1. *If* $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *converges for* $s = s_0$*, then it converges uniformly in every domain of the form*

$$\{s : \Re(s - s_0) \geq 0, |Arg(s - s_0)| \leq \theta\}$$

with $\theta \leq \frac{\pi}{2}$

Proof. We start by stating the following lemma:

Lemma 0.2. *Let* $0 < c < d \in \mathbb{R}$*. Then* $|e^{-cs} - e^{-ds}| \leq \frac{|s|}{\Re(s)}(e^{-c\Re(s)} - e^{-d\Re(s)})$.

Proof. We have

$$e^{-cs} - e^{-ds} = s \int_c^d e^{-ts} dt$$

Taking absolute values yield

$$|e^{-cs} - e^{-ds}| = |s \int_c^d e^{-ts} dt| \leq |s| \int_c^d |e^{-ts}| dt = |s| \int_c^d e^{-t\Re(s)} dt = \frac{|s|}{\Re(s)}(e^{-c\Re(s)} - e^{-d\Re(s)})$$

□

We prove the statement:
If $g(s)$ converges at $s = s_0$, then $f(s) := g(s - s_0)$ converges at $s = 0$, and so in everything that follows, we work with $s = s_0$.
Then, $f(0) = \sum_{i=1}^{\infty} a_n$ converges uniformly. We remark that we now wish to prove the uniform convergence of $f(s)$ on any domain

$$\{s : \Re(s) \geq 0, | \operatorname{Arg}(s) |\} \leq \theta$$

for some $\theta < \frac{\pi}{2}$.
We write $s = x + iy$ with $x, y \in \mathbb{R}$. For every s in the above defined domain, we have $x \geq 0$ and so $\operatorname{Arg}(s) = \arctan(\frac{y}{x})$. The second condition thus reads

$$| \operatorname{Arg}(s) | \leq \theta \iff | y | \leq x \tan(\theta)$$

We finish by writing $|s|^2 = x^2 + y^2 \leq x^2 + x^2 \tan(\theta)^2 = x^2(1 + \tan(\theta))^2$. Therefore, the condition on the argument is equivalent to the condition $|s| \leq xM$, with $M = \sqrt{1 + \tan(\theta)}$ and so $\frac{|s|}{\Re(s)} \leq M$ with any $M \geq 0$. Effectively, it remains to show the result on

$$\{s : \Re(s) \geq 0, \frac{|s|}{\Re(s)} \leq M\}$$

As in Abel's Lemma, we introduce for $r, m \in \mathbb{N}, r > m$:

$$A_{m,r} = \sum_{n=m}^{r} a_n$$

By the uniform convergence of $f(0)$, we deduce that there exists $N \in \mathbb{N}$ large enough such that for all $r > m \geq N$, we have $|A_{m,r}| \leq \varepsilon$. We now set $b_n = \frac{1}{n^s}$ (such that $a_n b_n$ is exactly the $n-$th term of the Dirichlet series) to obtain from Abel's Lemma

$$|S_{m,r}| = |\sum_{n=m}^{r-1} A_{m,n}(n^{-s} - (n+1)^{-s}) + A_{m,r}b_r|$$

Using the above lemma with $c = \ln(n), d = \ln(n+1)$ as well as the bound on $|A_{m,r}|$, we conclude

$$\begin{aligned}
|S_{m,r}| &\leq |\sum_{n=m}^{r-1} A_{m,n}(n^{-s} - (n+1)^{-s})| + |A_{m,r}r^{-s}| \\
&\leq \varepsilon \sum_{n=m}^{r-1} \frac{|s|}{\Re(s)}(n^{-\Re(s)} - (n+1)^{-\Re(s)}) + \varepsilon \\
&\leq \varepsilon(1 + \frac{|s|}{\Re(s)}(m^{-\Re(s)} - r^{-\Re(s)}) \\
&\leq \varepsilon(1 + M)
\end{aligned}$$

This shows that $|S_{m,r}| \xrightarrow{m\to\infty} 0$ (in fact $S_{m,r} \to 0$ uniformly on the desired domain) and since $S_{m,r}$ is the expression of the difference of the partial sums, we conclude that the series converges uniformly for all s satisfying $\Re(s) \geq 0$ and $\frac{|s|}{\Re(s)} \leq M$.
Undoing the original translation by s_0, we conclude the proof.

□

3. Show that

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{\gamma_n}{n^s}, \quad \gamma_n = \#\{\mathfrak{a} : N\mathfrak{a} = n\}$$

is absolutely convergent for $\Re(s) > 1$.

Proof. We will prove that the Euler Product

$$\prod_{\mathfrak{p}} (1 - N\mathfrak{p}^{-s})^{-1}$$

is absolutely convergent for $\Re(s) > 1$ and will then rearrange this product to prove that it is in fact the Euler product of $\zeta_K(s)$. To start with, we let $s \in \mathbb{R}$, as the previous exercise implies the desired result.
An infinite product of the form $\prod_i (1+x_n)$ with $|x_i| < 1$ is convergent if the series $\sum_i x_i$ converges.(We omit the proof, but the idea is the monotone convergence theorem and an upper bound on the product given by the exponential of the sum.)
In our case, we have $x_i = \frac{1}{N\mathfrak{p}_i^s}$ and so we wish to understand the sum

$$\sum_{\mathfrak{p}} \frac{1}{N\mathfrak{p}^s}.$$

For any prime $\mathfrak{p}$ lying over $p = \mathfrak{p} \cap \mathbb{Z}$, the norm satisfies $N\mathfrak{p} \geq p$, and therefore $\frac{1}{p^s} \geq \frac{1}{N\mathfrak{p}^s}$. Furthermore, there is a maximum of $[K : \mathbb{Q}]$ primes $\mathfrak{p}$ lying over a common p, which gives us

$$\sum_{\mathfrak{p}} \frac{1}{N\mathfrak{p}^s} \leq \sum_{p} \frac{[K : \mathbb{Q}]}{p^s} < \infty$$

and the right hand side is a convergent series, since it is bounded by the zeta function at $s > 1$ which is convergent. This shows convergence for all $s \in \mathbb{C}$ satisfying $\Re(s) > 1$.
To retrieve the Zeta function from its product, the heuristic is as follows: We recall that the Taylor expansion of $(1-x)^{-1}$ is precisely $\sum_n x^n$, and so upon taking the product of, say the primes $\mathfrak{p}$ with norms smaller than N, we get

$$\begin{aligned} \prod_{\mathfrak{p},\ N\mathfrak{p} \leq N} (1 - N\mathfrak{p}^{-s})^{-1} &= \prod_{\mathfrak{p}, N\mathfrak{p} \leq N} \left(1 + \frac{1}{N\mathfrak{p}^s} + \frac{1}{N\mathfrak{p}^{2s}} + \ldots\right) \\ &= \sum_{m_1, \ldots, m_k \geq 0} \frac{1}{(N\mathfrak{p}_1^{m_1} \ldots N\mathfrak{p}_k^{m_k})^s} \end{aligned}$$

and we recognize that by unique factorization into primes, the last sum is precisely the sum $\sum_{\mathfrak{a}} N\mathfrak{a}^{-s}$ over the integral ideals $\mathfrak{a}$.

□

4. Let $p, m \in \mathbb{N}$ with p prime such that $p \nmid m$. Our goal is to prove that

$$(1 - T^f)^{\varphi(m)/f} = \prod_{\chi \in \widehat{(\mathbb{Z}/m\mathbb{Z})^\times}} (1 - \chi(p)T) \quad \forall T \in \mathbb{C}$$

where $\varphi(m) =| (\mathbb{Z}/m\mathbb{Z})^\times |$ and f is the order of p in $(\mathbb{Z}/m\mathbb{Z})^\times$.

(a) Let $K = \mathbb{Q}(\zeta_m), G = \mathrm{Gal}(K/\mathbb{Q})$ and $Z = Z(p)$ the decomposition group of p. Show that the restriction map $\widehat{G} \to \widehat{Z}$ is surjective.

Proof. We prove the general statement that given G an abelian group and $H < G$, the restriction map $\widehat{G} \to \widehat{H}$ is surjective. It is easy to see that this map is a group homomorphism, and has kernel $H^\perp$, hence we obtain an injective homomorphism $\frac{\widehat{G}}{H^\perp} \to \widehat{H}$. Finally, counting orders, we get that it is necessarily an isomorphism which implies that the restriction is surjective.
We finish by recalling that $Z < G$ and $G \simeq \left(\frac{\mathbb{Z}}{m\mathbb{Z}}\right)^\times$ is an abelian group.

□

(b) Deduce that

$$\prod_{\chi \in \widehat{G}} (1 - \chi(p)T) = \prod_{\psi \in \widehat{Z}} (1 - \psi(p)T)^g$$

where $g = \# \ker(\widehat{G} \to \widehat{Z}) = \varphi(m)/f$.

Proof. We use (a) to split the product as

$$\prod_{\chi \in \widehat{G}} (1 - \chi(p)T) = \prod_{\psi \in \widehat{Z}} \prod_{\substack{\chi \in \widehat{G} \\ \chi_{|Z} = \psi}} (1 - \chi(p)T)$$

Finally, we note that $\chi(p)$ depends only on the image of χ in $\widehat{Z}$ (via the identification $Z \simeq X/Y_1$) and we may simplify the product to obtain $\prod_{\psi \in \widehat{Z}} (1 - \psi(p)T)^{a_\psi}$. The powers a_ψ that appear in the product is then exactly the cardinality of a coset of $\widehat{Z}$ in $\widehat{G}$ which is given by

$$\begin{aligned} \#\chi\widehat{Z} &= \# \ker(\widehat{G} \to \widehat{Z}) \\ &= \frac{\#\widehat{G}}{\#\widehat{Z}} \\ &= \frac{\varphi(m)}{f} = g \end{aligned}$$

Therefore,

$$\prod_{\chi \in \widehat{G}} (1 - \chi(p)T) = \prod_{\psi \in \widehat{Z}} (1 - \psi(p)T)^g.$$

□

(c) Identify $\widehat{Z} \simeq \mu_f$ via $\psi \mapsto \psi(p)$ and use

$$\prod_{\eta \in \mu_f} (1 - \eta T) = 1 - T^f$$

to conclude.

Proof. Using the stated result and combining with our result from (c), we obtain from the identification $\widehat{Z} \simeq \mu_f$:

$$\begin{aligned}\prod_{\chi\in\widehat{G}}(1-\chi(p)T) &= \prod_{\psi\in\widehat{Z}}(1-\psi(p)T)^g \\ &= \prod_{\eta\in\mu_f}(1-\eta T)^g \\ &= (1-T^f)^g = (1-T^f)^{\varphi(m)/f}\end{aligned}$$

as desired.

□

Proof. We prove

$$\prod_{\eta\in\mu_f}(1-\eta T) = 1-T^f$$

The classical cyclotomic polynomial reads as $X^f - 1 = \prod_\eta (X-\eta)$. Upon doing the change of variable $X = 1/T$, we obtain $T^{-f} - 1 = \prod_\eta (1/T - \eta)$ hence after multiplying by T^f

$$1 - T^f = \prod_\eta (1-\eta T)$$

□